13 Functional series. Uniform convergence

For Augustin-Louis Cauchy the main starting point in the analysis of holomorphic functions was differentiability, and as a consequence, the integral property. Bernhard Riemann has a much more geometric point of view and, along with differentiability, started with the property of being conformal. There was yet another point of view, in particular promoted by Karl Weierstrass (1815–1897), who looked at the complex analysis from the point of view that "nice" functions are those that can be represented as convergent power series. In this lecture I present the necessary background with the ultimate goal to see how it is possible to obtain one more characterization of holomorphic functions.

13.1 Functional series and sequences

In one of the previous lectures I already introduced complex sequences and series and discussed a little their convergence, mainly referring to your Calc II experience. I needed it to rigorously define complex exponent and deduce its properties. Here I will be more detailed.

I will be studying the series of the form

$$g_0(z) + g_1(z) + \ldots + g_n(z) + \ldots, \tag{13.1}$$

where

$$g_n\colon E \longrightarrow \mathbf{C},$$

are the functions defined on the same domain E. My goal to make sense of the expression $g_0(z) + g_1(z) + \ldots + g_n(z) + \ldots = g(z)$ for some $g\colon E \longrightarrow \mathbf{C}$.

Together with the series (13.1) I consider a sequence of functions $(f_n)_{n=0}^{\infty}$, $f_n\colon E \longrightarrow \mathbf{C}$, where each f_n is the partial sum of (13.1):

$$f_n(z) = g_0(z) + \ldots + g_n(z).$$

In general, *sequences* and *series* are the same object looked at from two different perspectives; specifically, given a series of the form (13.1), I can always consider its sequence of partial sums, in the opposite direction, if I deal with a sequence (f_n) I can construct the series (13.1) by setting $g_0(z) = f_0(z)$, $g_n(z) = f_n(z) - f_{n-1}(z)$. So all the definitions below can be stated both in the language of sequences and series, please carefully check to which object a given definition applies.

Definition 13.1. *Let $(f_n)_{n=0}^{\infty}$ be a sequence of functions, $f_n\colon E \longrightarrow \mathbf{C}$ for all n. It is said that this sequence converges pointwise to the function $f\colon E \longrightarrow \mathbf{C}$ if for any $\epsilon > 0$ and any $z \in E$ there is a natural number $N \in \mathbf{N}$ (which can depend on both ϵ and z), such that for all $n \geq N$*

$$|f_n(z) - f(z)| < \epsilon.$$

Similarly,

Definition 13.2. *Series* (13.1) *converges pointwise to the function $f\colon E \longrightarrow$ if for any $\epsilon > 0$ and $z \in E$ there exists $N \in \mathbf{N}$, such that*

$$|g_0(z) + g_1(z) + \ldots + g_n(z) - f(z)| < \epsilon$$

for all $n \geq N$.

Math 452: *Complex Analysis* by Artem Novozhilov©
e-mail: artem.novozhilov@ndsu.edu. Spring 2019

So the notion of pointwise convergence is simply carrying to the functional series the definition of convergence of numerical series: fix z, get a numerical series, discuss the convergence. We already had this example, let me recall it again.

Example 13.3. Let $f_n(z) = z^n$. Then the sequence (f_n) converges pointwise to 0 if $|z| < 1$, diverges to ∞ if $|z| > 1$, converges to 1 if $z = 1$, and has no limit if $|z| = 1, z \neq 1$.

Now I can consider my fundamental example for series.

Example 13.4 (Geometric series)**.** The series of the form

$$1 + z + z^2 + \ldots + z^n + \ldots$$

is called *geometric*. I claim that this series converges pointwise to $1/(1 - z)$ if $|z| < 1$ and diverges if $|z| \geq 1$.

Here I get this conclusion by explicitly considering the sequence of partial sums:

$$\begin{aligned}
f_0(z) &= 1,\\
f_1(z) &= 1 + z,\\
f_2(z) &= 1 + z + z^2,\\
&\ldots\\
f_n(z) &= 1 + z + \ldots + z^n = \frac{1 - z^{n+1}}{1 - z}, \quad z \neq 1,
\end{aligned}$$

where the last formula can be proved by multiplying both sides of equality by $1 - z$. Now

$$\left| f_n(z) - \frac{1}{1 - z} \right| = \frac{|z|^{n+1}}{|1 - z|}.$$

I have that if $|z| < 1$ then taking $N = \left\lfloor \frac{\ln(\epsilon|1-z|)}{\ln|z|} \right\rfloor$ does the rick to prove the pointwise convergence. If $z = 1$ my series is clearly divergent, if $|z| > 1$ $|z|^n$ goes to infinity and hence the series diverge, finally, if $|z| = 1, z \neq 1$, z^n travels on the unit circle and hence there is no limit.

Once again, to emphasize its importance,

$$1 + z + z^2 + \ldots + z^n + \ldots = \frac{1}{1 - z}, \quad |z| < 1,$$

where the convergence understood pointwise.

Recall that I talked about convergence of numerical series, say $c_0 + c_1 + \ldots + c_n + \ldots$, I also discussed the notion of *absolute* convergence. Namely, series $\sum_{n=0}^{\infty} c_n$ converges absolutely if the series $\sum_{n=0}^{\infty} |c_n|$ with *non-negative* terms converges. Absolute convergence is a more stringent property than simple convergence.

Proposition 13.5. *An absolutely convergent series converges.*

Proof. ∎

The converse is not true, i.e., there are series that converge but without absolute convergence. Such series are called *conditionally* convergent and they have a somewhat strange nonintuitive behavior; e.g., by rearranging the terms of a series I can guarantee that the sum will be equal to *any* specified number. In order not to deal with such strange behavior, I will be discussing only absolutely convergent series.

Example 13.6. The geometric series converges absolutely to $1/(1-z)$ for any $|z| < 1$. Indeed, if $|z| < 1$ then, to test the absolute convergence, I consider

$$1 + |z| + |z^2| + \ldots = 1 + |z| + |z|^2 + \ldots = 1 + r + r^2 + \ldots = \frac{1}{1-r},$$

where $r = |z|$, and therefore for each fixed $z \in (0, 1)$ the series converges absolutely.

13.2 Uniform convergence

As natural as it may seem, pointwise convergence is a complicated form of convergence. Instead, I will concentrate on something more natural (the meaning of this "natural" will be clear later).

Definition 13.7. *Let $(f_n)_{n=0}^{\infty}$, $f_n \colon E \longrightarrow \mathbf{C}$ be a sequence of functions. It is said it converges uniformly to $f \colon E \longrightarrow \mathbf{C}$ if for any $\epsilon > 0$ there exists a natural number $N \in \mathbf{N}$ (which may depend on ϵ), such that*

$$|f_n(z) - f(z)| < \epsilon$$

for all $n \geq N$ and all $z \in E$.

The key difference with the pointwise convergence is that now ϵ does not depend on a particular point z, it must work *uniformly* for all possible z. This observation indicates that if a sequence converges uniformly, it converges pointwise, but the converse may not be true.

Example 13.8. The sequence (f_n), $f_n(z) = z^n$ does not converge uniformly on $|z| < 1$. Let me prove this fact by contradiction. Assume that it does converge to 0 uniformly on the whole disk $|z| < 1$. That is, for arbitrary ϵ I can find N, such that

$$|z^n - 0| = |z^n| = |z|^n < \epsilon$$

for all $n \geq N$ and all $z \in B(0, 1)$. On another hand, I have, since r^N is a continuous function with fixed N, that

$$\lim_{|z| \to 1-0} |z|^N = \lim_{r \to 1-0} r^N = 1.$$

That is, for each fixed ϵ I am able to find z with $|z| < 1$ and $|z|^N$ as close to 1 as I wish, which contradicts the inequality $r^N < \epsilon$ for ϵ strictly less than 1.

However, if I lighten my condition a little, I can show that sequence (z^n) converges uniformly to $1/(1-z)$ for any $|z| \in [0, r)$, where $r < 1$ and fixed. Indeed, fix $r \in [0, 1)$ and $\epsilon > 0$. Note that $|z| \leq r$ then $\ln |z| \leq \ln |r|$ and $1/\ln |z| \geq 1/\ln r$. Take $N = \lfloor \frac{\ln \epsilon}{\ln r} \rfloor + 1$. Due to the discussed inequalities for any z such that $|z| \leq r$ it will be true that

$$|z|^n < \epsilon,$$

for any $n \geq N$, which proves the uniform convergence.

Similarly,

Definition 13.9. *Series* (13.1) *converges uniformly to* f *if for any* $\epsilon > 0$ *there is* $N \in \mathbf{N}$ *such that*

$$|g_0(z) + g_1(z) + \ldots + g_n(z) - f(z)| < \epsilon$$

for all $n \geq N$ *and all* $z \in E$.

Example 13.10. As it should be intuitively expected the geometric series does not converge uniformly on $|z| < 1$. However, it does converge uniformly in any ball $B(0, r)$ with $r < 1$ fixed. The details are left as an exercise.

So, what is so special about the uniform convergence? Here are the main points.

Proposition 13.11. *Let* (f_n) *converge uniformly on* E *to* f. *If* f_n *are continuous on* E *so is* f.

Proof. So, I need to show that for any fixed $z_0 \in E$ and any given $\epsilon > 0$ there is $\delta > 0$ such that

$$|f(z) - f(z_0)| < \epsilon$$

if $|z - z_0| < \delta$, assuming that all such $z \in E$.

Now

$$\begin{aligned} |f(z) - f(z_0)| = |f(z) - f(z_0) + f_n(z) - f_n(z) + f_n(z_0) - f_n(z_0)| \leq \\ \leq |f(z) - f_n(z)| + |f(z_0) - f_n(z_0)| + |f_n(z) - f_n(z_0)| = \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon, \end{aligned}$$

where first two terms are less than epsilon due to uniform convergence, and the third due to continuity of f_n. ∎

Remark 13.12. This proposition sometimes allows to check easily that convergence is not uniform. For instance, consider a sequence $(x^n), x \in [0, 1]$. This sequence converges pointwise on this interval to the function that is equal to 0 if $0 \leq x < 1$ and to 1 if $x = 1$, which is a discontinuous function. Therefore, the convergence cannot be uniform (note that all x^n are continuous).

Corollary 13.13. *If* (13.1) *converges uniformly to* f *on* E *and all* g_n *are continuous so is* f.

Remark 13.14. In essence, uniform convergence of a series with continuous terms means that one can interchange the limits (recall that f is continuous at z_0 if $\lim_{z \to z_0} f(z) = f(\lim_{z \to z_0} z) = f(z_0)$):

$$f(z_0) = \lim_{z \to z_0} f(z) = \lim_{z \to z_0} \lim_{n \to \infty} f_n(z) = \lim_{n \to \infty} \lim_{z \to z_0} f_n(z) = \lim_{n \to \infty} f_n(\lim_{z \to z_0} z) = \lim_{n \to \infty} f_n(z_0) = f(z_0).$$

Since operations of differentiation and integration are limits, it indicates that for the series that converge uniformly, these operations can be performed termwise.

Proposition 13.15. *Let* (f_n) *converge to* f *uniformly on* E, f_n *are continuous, and* $\gamma \subseteq E$ *is a path. Then*

$$\lim_{n \to \infty} \int_\gamma f_n = \int_\gamma \lim_{n \to \infty} f_n = \int_\gamma f.$$

Proof. By recalling the ML–inequality the proof is immediate:

$$\left|\int_\gamma f_n - \int_\gamma f\right| = \left|\int_\gamma (f_n - f)\right| \leq \max_{z\in\gamma} |f_n(z) - f(z)| L,$$

where L is the length of γ. Since convergence is uniform, it is true that for sufficiently large n and all $z \in E$

$$|f_n(z) - f(z)| < \frac{\epsilon}{L},$$

therefore

$$\left|\int_\gamma f_n - \int_\gamma f\right| < \epsilon,$$

which concludes the proof. ■

Corollary 13.16. *Let* (13.1) *converge uniformly to f on E and $\gamma \subseteq E$ then*

$$\int_\gamma f = \int_\gamma \sum g_n = \sum \int_\gamma g_n,$$

i.e., for a uniformly convergent series one is allowed to interchange the order of integration and summation.

Finally, sometimes one needs a simple condition to check uniform convergence. Here is one of the most useful criteria.

Proposition 13.17 (Weierstrass M-test)**.** *Consider* (13.1) *and assume that $|g_k(z)| \leq M_k$ for some constants M_k. Moreover, assume that $\sum_{k=0}^{\infty} M_k$ converges. Then both $\sum_{k=0}^{\infty} |g_k(z)|$ and $\sum_{k=0}^{\infty} g_k(z)$ converge uniformly on E.*

Remark 13.18. In this case it is said, a little confusingly, that the series converges absolutely and uniformly. The confusion stems from the fact that if a series converges absolutely, and the same converges uniformly, we still cannot conclude that the series converges absolutely and uniformly as expected. But these subtle points will not bother us here.

Proof. See the textbook. ■